Paraxial, Thin-Lens Analysis of Fixed-Tune, Non-Scaling FFAs with Two Magnets per Cell

Stephen Brooks

May 6, 2022

1 Introduction

Fixed-field accelerators (FFAs) have closed orbits that change as a function of beam momentum. It is sometimes useful to avoid various resonances by keeping the tunes constant as these orbits change. A well-known example is the *scaling FFA* where the entire beam orbit and optics are geometrically scaled as a function of momentum. However, this is stricter than necessary: cells with three or more lenses can have fixed tunes even when the focussing strengths in the lenses change [1].

Recently, Dejan Trbojevic has found a pair of nonlinear magnets that produce fixed tunes *and fixed beta functions* while not being a scaling FFA [2]. This improves on a more approximate solution in [3]. Notably a scaling FFA would require one magnet to be entirely reverse-bending, whereas the Trbojevic solution does not do this and instead resembles an intermediate point between the traditional scaling field profiles and the nonscaling profiles centred around a momentum with equal and positive fields.

This suggests there are at least three levels of stringency that one can apply to a fixed-tune FFA design:

1. Fixed cell tunes (in both planes) as a function of momentum;
2. Fixed optics (beta functions) as a function of momentum;
3. Similarity of all orbits via a scaling symmetry law $\Leftrightarrow$ traditional scaling FFA.

This note studies the interesting case #2 above (#3 being fully characterised by the orbit at a single energy) in the simplest possible example: a cell of two thin lenses in the small angle (paraxial) approximation.

2 Definitions of Variable Functions

The cell consists of an F magnet, a drift of length d_1, a D magnet and a drift of length d_2. The beam position and angle in these magnets and drifts, as a function of momentum p, are

$$x_F(p), \quad x'_1(p), \quad x_D(p), \quad x'_2(p),$$

respectively. The bending angle in each thin lens is $\theta_e(p)$ for $e = F, D$ and the normalised integrated field is $b_e(x)$, where these are related by

$$\theta_e(p) = \frac{b_e(x_e(p))}{p}.$$

This is a total of eight functions, which need as many constraints.

3 Constraints

There are a total of eight functional constraints: 2+4+2 as given in the subsections below.

3.1 Constant Normalised Gradient

It appears in the two-magnet-per-cell case, the normalised gradient in each magnet and the beta functions remain constant. Thus,

$$b'_e(x_e(p)) = k_e p,$$

for $e = F, D$ and two constants k_F, k_D, gives two constraints.

3.2 Paraxial Dynamics and Closure

Starting with angles, it is clear that

$$x'_2(p) = x'_1(p) + \theta_D(p)$$

but going back to x'_1 involves wrapping to the next cell, so the overal constant cell angle θ_{cell} must be subtracted:

$$x'_1(p) = x'_2(p) + \theta_F(p) - \theta_{\text{cell}}.$$

For the positions, paraxial tracking through the drifts gives

$$x_D(p) = x_F(p) + d_1 x'_1(p)$$

and wrapping to the start of the cell,

$$x_F(p) = x_D(p) + d_2 x'_2(p) - \Delta x_{\text{cell}},$$

where a constant transverse displacement Δx_{cell} has also been allowed.

3.3 Relation of Angles to Normalised Integrated Fields

These two constraints are the ones from the previous section:

$$\theta_e(p) = \frac{b_e(x_e(p))}{p},$$

for $e = F, D$.

4 Elimination

With eight functions and eight functional constraints, all variable functions should be eliminated, leaving only some constants, some in the form of boundary conditions for the differential equation part.

4.1 Linear Part

Four of the equations are linear (in x_e, θ_e, x'_n) with additive constants and allow some elimination. Using the equations for x'_1 and x'_2,

$$\begin{aligned} x'_1(p) &= x'_2(p) + \theta_F(p) - \theta_{\text{cell}} \\ &= x'_1(p) + \theta_D(p) + \theta_F(p) - \theta_{\text{cell}} \\ \Rightarrow \quad \theta_D(p) + \theta_F(p) &= \theta_{\text{cell}}. \end{aligned}$$

This is common sense so far (the lens angles added together equal the cell angle) and can be used to eliminate θ_D.

Using the equations for x_F and x_D,

$$\begin{aligned} x_F(p) &= x_D(p) + d_2 x'_2(p) - \Delta x_{\text{cell}}, \\ &= x_F(p) + d_1 x'_1(p) + d_2 x'_2(p) - \Delta x_{\text{cell}}, \\ \Rightarrow \quad d_1 x'_1(p) + d_2 x'_2(p) &= \Delta x_{\text{cell}}. \end{aligned}$$

Again, this is a common sense evaluation of the transverse offset in the cell and can be used to eliminate x'_2.

Using the elimination for x'_2 in in the x'_1 equation gives:

$$\begin{aligned} x'_1(p) &= x'_2(p) + \theta_F(p) - \theta_{\text{cell}} \\ &= \frac{\Delta x_{\text{cell}} - d_1 x'_1(p)}{d_2} + \theta_F(p) - \theta_{\text{cell}} \\ \Rightarrow \quad \left(1 + \frac{d_1}{d_2}\right) x'_1(p) &= \frac{\Delta x_{\text{cell}}}{d_2} + \theta_F(p) - \theta_{\text{cell}}, \end{aligned}$$

which can be used to eliminate x'_1. This can immediately be used in the x_D equation

$$\begin{aligned} x_D(p) &= x_F(p) + d_1 x'_1(p) \\ &= x_F(p) + \frac{d_1}{1 + \frac{d_1}{d_2}} \left(\frac{\Delta x_{\text{cell}}}{d_2} + \theta_F(p) - \theta_{\text{cell}} \right) \\ &= x_F(p) + \frac{d_1 d_2}{d_1 + d_2} \left(\theta_F(p) + \frac{\Delta x_{\text{cell}}}{d_2} - \theta_{\text{cell}} \right), \end{aligned}$$

which eliminates x_D by expressing it in terms of x_F and θ_F, which are the only remaining functional variables apart from the b_e which weren't in the linear part.

4.2 Geometrisation

To eliminate the b_e functions and work entirely in terms of angles, note that

$$\theta_e(p) = \frac{b_e(x_e(p))}{p} \quad \Rightarrow \quad p\theta_e(p) = b_e(x_e(p))$$

and the term on the right now looks superficially similar to the $b_e'(x_e(p))$ appearing in the 'constant normalised gradient' equation. Taking the derivative of both sides with respect to p,

$$\begin{aligned}\theta_e(p) + p\theta_e'(p) &= b_e'(x_e(p))x_e'(p) \\ &= k_e p x_e'(p).\end{aligned}$$

Taking care to note that x_e' is a p derivative of x_e, not an angle like x_n', this has eliminated the b_e variables.

Including the elimination of θ_D and x_D, the two equations in this part are now

$$\theta_F(p) + p\theta_F'(p) = k_F p x_F'(p)$$

and

$$\begin{aligned}\theta_{\text{cell}} - \theta_F(p) - p\theta_F'(p) &= k_D p x_D'(p) \\ &= k_D p \left(x_F'(p) + \frac{d_1 d_2}{d_1 + d_2}\theta_F'(p) \right).\end{aligned}$$

4.3 Solution for $\theta_F(p)$

x_F and θ_F are the only functions left and the above are the only two constraints left. The x_F' terms can be cancelled by taking k_D times the first equation minus k_F times the second one:

$$\begin{aligned}& k_D(\theta_F(p) + p\theta_F'(p)) - k_F(\theta_{\text{cell}} - \theta_F(p) - p\theta_F'(p)) \\ =\ & k_D k_F p x_F'(p) - k_F k_D p \left(x_F'(p) + \frac{d_1 d_2}{d_1 + d_2}\theta_F'(p) \right) \\ =\ & -k_F k_D p \frac{d_1 d_2}{d_1 + d_2}\theta_F'(p),\end{aligned}$$

$$\Rightarrow \quad (k_D + k_F)\theta_F(p) - k_F\theta_{\text{cell}} = \left(-k_D - k_F - k_F k_D \frac{d_1 d_2}{d_1 + d_2} \right) p\theta_F'(p),$$

which is a first order differential equation for $\theta_F(p)$ and just needs an initial condition $\theta_F(p_0) = \theta_{F0}$. It is of the form

$$f'(x) = \frac{A}{x}f(x) + \frac{B}{x},$$

which has general solution

$$f(x) = Cx^A - \frac{B}{A}.$$

Here, we put

$$\begin{aligned}A &= \frac{k_D + k_F}{-k_D - k_F - k_F k_D \frac{d_1 d_2}{d_1 + d_2}} \\ B &= \frac{-k_F\theta_{\text{cell}}}{-k_D - k_F - k_F k_D \frac{d_1 d_2}{d_1 + d_2}}\end{aligned}$$

and C such that

$$\theta_F(p_0) = Cp_0^A - \frac{B}{A} = \theta_{F0} \quad \Rightarrow \quad C = \frac{\theta_{F0} + \frac{B}{A}}{p_0^A}$$

with solution

$$\theta_F(p) = Cp^A - \frac{B}{A}.$$

4.4 Solution for $x_F(p)$

Substituting the formula for θ_F above into this equation

$$\theta_F(p) + p\theta_F'(p) = k_F p x_F'(p)$$

gives:

$$Cp^A - \frac{B}{A} + p(ACp^{A-1}) = (1+A)Cp^A - \frac{B}{A} = k_F p x_F'(p)$$
$$\Rightarrow \quad x_F'(p) = \frac{(1+A)C}{k_F} p^{A-1} - \frac{B}{Ak_F}\frac{1}{p}.$$

With an initial condition $x_F(p_0) = x_{F0}$, this gives the orbit positions as

$$x_F(p) = \frac{(1+A)C}{Ak_F} p^A - \frac{B}{Ak_F}\ln p + \left(x_{F0} - \frac{(1+A)C}{Ak_F} p_0^A + \frac{B}{Ak_F}\ln p_0\right).$$

5 Magnetic Field

The magnetic field in the F magnet can be found by substituting the explicit formulae for $x_F(p)$ and $\theta_F(p)$ in the previous section into

$$b_F(x_F(p)) = p\theta_F(p),$$

however it is doubtful that the function $x_F(p)$ can be inverted to give b_F analytically as a function of x, so this 'parametric' form should suffice. A similar thing is possible for the D magnet with more substitutions.

An exceptional case is where $C = 0$ and Scott Berg pointed out that θ_F is constant as in the original scaling FFA. This gives a logarithmic $x_F(p)$ and an exponential $b_F(x)$: it is the 'straight' small-angle limit of the scaling FFA, which is already known to have an exponential field dependence [4].

6 Note

I had a previous attempt at deriving a solution for fixed-tune FFAs in [5] but the equations produced, while in theory solvable mechanically, were so complex I never substituted anything in to them to check. The assumption of fixed *optics* (via constant normalised gradient) rather than fixed tunes helped a lot here, so hopefully checking these equations will be easier.

References

[1] *Three-Lens Lattices for Extending the Energy Range of Non-scaling FFAGs*, S.J. Brooks, Proc. IPAC 2011, available at `https://accelconf.web.cern.ch/IPAC2011/papers/weps086.pdf`

[2] D. Trbojevic, to be published in Proc. IPAC 2022.

[3] *Permanent Magnets Future Electron Ion Colliders at RHIC and LHeC*, D. Trbojevic *et al.*, Proc. IPAC 2021, available at `https://accelconf.web.cern.ch/ipac2021/papers/tupab028.pdf`

[4] *Straight Section in Scaling FFAG Accelerator*, JB. Langrange and Y. Mori, Proc. PAC 2009, available at `https://accelconf.web.cern.ch/PAC2009/papers/fr5pfp002.pdf`

[5] *Fixing the Tune of Nonscaling FFAGs in the Thin Lens Paraxial Approximation*, S.J. Brooks (2010), available at `https://stephenbrooks.org/ap/report/2010-5/tunefix.pdf`