Trapping Two Dissimilar Rigidity Ions with Identical Average Dynamics

Stephen Brooks

January 10, 2025

1 Introduction

It may seem obvious that ions with different charge-to-mass ratios (q/m) would have different dynamics in a trap or periodic focussing system, leading to different equilibrium bunch shapes and sizes. However, the ponderomotive acceleration in an oscillating field is proportional to $(q/m)^2$ rather than q/m for the direct field, leading to the possibility of combining the two effects to cancel any differences in the average dynamics between two chosen rigidities.

2 Ponderomotive Acceleration

Nonrelativistically, the ponderomotive acceleration is given by

$$\mathbf{a}_p = -\frac{q^2}{4m^2\omega^2}\nabla|\mathbf{E}_p|^2,$$

for a oscillating electric field $\mathbf{E}(\mathbf{x},t) = \mathbf{E}_p(\mathbf{x})\sin(\omega t)$. Compare this with the direct acceleration

$$\mathbf{a}_d = \frac{q}{m}\mathbf{E}_d.$$

For two ion species, consider the case where

$$\nabla|\mathbf{E}_p|^2 = \frac{4\omega^2}{\frac{q_1}{m_1}+\frac{q_2}{m_2}}\mathbf{E}_d.$$

The total (average) acceleration for ion species i is then

$$\begin{aligned}
\mathbf{a}_i &= \frac{q_i}{m_i}\mathbf{E}_d - \frac{q_i^2}{4m_i^2\omega^2}\nabla|\mathbf{E}_p|^2 \\
&= \frac{q_i}{m_i}\mathbf{E}_d - \frac{q_i^2}{4m_i^2\omega^2}\frac{4\omega^2}{\frac{q_1}{m_1}+\frac{q_2}{m_2}}\mathbf{E}_d \\
&= \left(\frac{q_i}{m_i} - \frac{q_i^2}{m_i^2}\frac{1}{\frac{q_1}{m_1}+\frac{q_2}{m_2}}\right)\mathbf{E}_d \\
&= \frac{\frac{q_i}{m_i}\frac{q_1}{m_1}+\frac{q_i}{m_i}\frac{q_2}{m_2}-\frac{q_i^2}{m_i^2}}{\frac{q_1}{m_1}+\frac{q_2}{m_2}}\mathbf{E}_d.
\end{aligned}$$

One of the terms on the numerator is also $\frac{q_i^2}{m_i^2}$, so it cancels, leaving only the mixed term. The average acceleration is therefore the same for both ions:

$$\mathbf{a}_i = \frac{\frac{q_1}{m_1}\frac{q_2}{m_2}}{\frac{q_1}{m_1} + \frac{q_2}{m_2}}\mathbf{E}_d.$$

3 Quadrupole Field

Consider the direct field $\mathbf{E}_d = (kx, ky, -2kz)$. It has the potential $V_d = -\frac{k}{2}x^2 - \frac{k}{2}y^2 + kz^2$ such that $\mathbf{E}_d = -\nabla V_d$. For equal acceleration, it is required that

$$\nabla|\mathbf{E}_p|^2 = \frac{4\omega^2}{\frac{q_1}{m_1} + \frac{q_2}{m_2}}\mathbf{E}_d = \frac{-4\omega^2}{\frac{q_1}{m_1} + \frac{q_2}{m_2}}\nabla V_d.$$

This would be satisfied if

$$|\mathbf{E}_p|^2 = \frac{-4\omega^2}{\frac{q_1}{m_1} + \frac{q_2}{m_2}}V_d + C,$$

for any constant C. The solution cannot fill all of space as $|\mathbf{E}_p|^2 \geq 0$ and V_d can attain unbounded positive and negative values. C should generally be positive so that the region around the origin is feasible.

Linear electric fields cannot attain the required saddle shape of $|\mathbf{E}_p|^2$, so consider a combined dipole+sextupole electric field with

$$\mathbf{E}_p(0, 0, z) = (0, 0, \sqrt{C} - sz^2)$$

on the z axis. This extends symmetrically about the z axis to the full sextupole field

$$\mathbf{E}_p = \left(sxz, syz, \sqrt{C} + \frac{s}{2}x^2 + \frac{s}{2}y^2 - sz^2\right)$$

that satsifies Maxwell's equations in free space. Its modulus squared is

$$|\mathbf{E}_p|^2 = C + s\sqrt{C}(x^2 + y^2 - 2z^2) + \text{terms in } (x, y, z)^4.$$

Ignoring the higher order terms that are small near the origin, this has the correct form. Equating coefficients of $x^2 + y^2 - 2z^2$ gives

$$s\sqrt{C} = \frac{-4\omega^2}{\frac{q_1}{m_1} + \frac{q_2}{m_2}}\left(-\frac{k}{2}\right) \qquad \Rightarrow \qquad s = \frac{2k\omega^2}{\sqrt{C}\left(\frac{q_1}{m_1} + \frac{q_2}{m_2}\right)}.$$

4 Alternating Quadrupole Trap

The field in the previous section gives the two ion species identical accelerations near the origin, but it is not a trap because the effective potential is saddle-shaped. This can be rectified by modulating the entire potential at a frequency $\omega_a \ll \omega$, yielding an alternating effective quadrupole that is overall focussing by the alternating gradient principle. This can be done by setting $k(t) = k_0 \sin(\omega_a t)$ in the previous section, which is a simple modulation of $\mathbf{E}_d$ but only changes the sextupole part of $\mathbf{E}_p$ and not the dipole. This idea uses the ponderomotive motion principle *twice* because alternating gradient focussing reduces to ponderomotive focussing for small phase advances.

5 Accelerator Beamlines

This method can also be used to transport two species of different rigidity through a particle accelerator with the same optics. In this case, a 2D effective quadrupole is formed by a rapidly alternating dipole+sextupole field, superimposed with the correct amount of constant quadrupole and then this combination is alternated more slowly to give alternating gradient focussing.